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# Deterministic 3D-Perturbations of Planar Incompressible Flow Lead to Stochasticity

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We show that the long-time behavior of the stationary incompressible flow in $R^3$, which is close to a planar one, under broad generic assumptions, is, in a sense, stochastic. This stochasticity is a result of instability of the corresponding planar flow near the saddle points of the stream function. The stochastic process which describes long-time evolution of the slow component of the motion is calculated.

**KEY WORDS:** Averaging principle; random perturbation; incompressible fluid; turbulence.

Consider a stationary 3D-motion of an incompressible fluid invariant with respect to shifts along axis $x_3$. Let the $x_3$-component of the velocity field $V(x)$, $x \in R^3$, be zero, so that $V(x_1, x_2, x_3) = (v_1(x_1, x_2), v_2(x_1, x_2), 0)$. Since the field $V(x)$ is divergence free, one can introduce the stream function $\psi(x_1, x_2)$ such that

$$(v_1(x_1, x_2), v_2(x_1, x_2)) = \overline{\nabla}\psi(x_1, x_2) = \left(\frac{\partial\psi(x_1, x_2)}{\partial x_2}, \frac{\partial\psi(x_1, x_2)}{\partial x_1}\right)$$

The flow corresponding to this velocity field is planar and is the same in each plane $x_3 = \text{const}$. Its trajectories are defined by the equations

$$\dot{X}_1(t) = \frac{\partial\psi}{\partial x_2}(X(t)), \qquad \dot{X}_2(t) = \frac{\partial\psi}{\partial x_1}(X(t)), \qquad \dot{X}_3(t) = 0. \tag{1}$$

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The trajectories are connected components of the level sets of $\psi(x_1, x_2)$ in the planes $x_3 = \text{const}$. Assume for simplicity that $\psi(x_1, x_2)$ is generic (has a finite number of critical points and they are non-degenerate), and $\lim \psi(x_1, x_2) = \infty$, as $|x_1| + |x_2| \to \infty$, so that all level sets are bounded. We will write equations (1) in the form $\dot{X}_t = \overline{\nabla}\psi(\dot{X}_t)$ putting the $x_3$-component of $\overline{\nabla}\psi$ equal to zero.

Consider now a 3D-perturbation of this planar motion preserving the incompressibility:

$$V^\varepsilon(x) = V(x) + \varepsilon\beta(x), \ x \in R^3,$$

where $\beta(x)$ is a smooth vector field, div $\beta(x) = 0$, $0 < \varepsilon \ll 1$.

Then the trajectories of the perturbed flow are governed by the equations

$$\dot{\tilde{X}}^\varepsilon(t) = \overline{\nabla}\psi(\tilde{X}^\varepsilon(t)) + \varepsilon\beta(\tilde{X}^\varepsilon(t))$$

If $x_3$-component of $\beta(x)$ is again equal to zero, the new motion is also planar, and for, $0 < \varepsilon \ll 1$, is close to the original motion on any time interval. But if $\beta_3(x) \neq 0$, the behavior of the perturbed flow can be essentially different from the non-perturbed flow. Moreover, long time behavior of $\tilde{X}^\varepsilon(t)$ can be, in a sense, stochastic in spite of the deterministic character of the original flow and perturbations.

Assume, first, that the stream function $\psi(x_1, x_2)$ has just one critical point, a minimum at a point $O \in R^2$, $\psi(O) = 0$. Then, since we assume that $\psi(x_1, x_2) \to \infty$ as $|x_1| + |x_2| \to \infty$, each non-perturbed trajectory is periodic with a period

$$T(z) = \oint_{C(z)} \frac{dl}{|\nabla\psi(x)|},$$

where $C(z) = \{(x_1, x_2) \in R^2 : \psi(x_1, x_2) = z\}$, $dl$ is the length element on $C(z)$, and $z = \psi(x_1, x_2)$ if the trajectory starts at $(x_1, x_2, x_3) \in R^3$. On any finite (independent of $\varepsilon$) time interval $[0, T]$, $\tilde{X}_t^\varepsilon$ converges uniformly to the non-perturbed motion. But on intervals of order $\varepsilon^{-1}$, $\varepsilon \downarrow 0$, the perturbed motion essentially deviates from the non-perturbed one.

To describe the motion on large time intervals, it is convenient to rescale the time: Put $X_t^\varepsilon = \tilde{X}^\varepsilon(t/\varepsilon)$. Then $X^\varepsilon(t)$ satisfies the equation

$$\dot{X}^\varepsilon(t) = \frac{1}{\varepsilon}\overline{\nabla}\psi(X^\varepsilon(t)) + \beta(X^\varepsilon(t)), \quad X^\varepsilon(0) = x \in R^3. \tag{2}$$

The perturbed motion has two components, the fast one, which is, roughly speaking, the motion along the non-perturbed trajectories, and the slow component which describes displacements in the directions transversal to the non-perturbed trajectories.

The fast motion can be characterized by the invariant density $M_z(x)$ on each periodic trajectory $C(z)$:

$$M_z(x) = \left( |\nabla\psi(x)| \oint_{C(z)} \frac{dl}{|\nabla\psi(x)|} \right)^{-1} = \frac{1}{T(z)\,|\nabla\psi(x)|}, \qquad x \in C(z).$$

The slow component can be described by the evolution of $Y_t^\varepsilon = (\psi(X_1^\varepsilon(t), X_2^\varepsilon(t)), X_3^\varepsilon(t)) \in [0,\infty) \times R^1$, if $\psi(x_1, x_2)$ has just one minimum at $O \in R^2$, $\lim_{|x_1|+|x_2| \to \infty} \psi(x_1, x_2) = \infty$ and $\psi(O) = 0$.

The standard averaging principle implies that the slow component $Y_t^\varepsilon$ converges uniformly on any time interval $[0, T]$ to the solution of equations

$$\dot\psi(t) = \frac{1}{T(\psi(t))} B(\psi(t), X_3(t)),$$

$$\dot X_3(t) = \frac{1}{T(\psi(t))} D(\psi(t), X_3(t)), \tag{3}$$

$$\psi(0) = \psi(x_1, x_2),\ x_3(0) = x_3.$$

The coefficients $B(z, x_3)$ and $D(z, x_3)$ are defined as follows:

$$B(z, x_3) = \oint_{C(z)} \frac{\beta(x)\cdot\nabla\psi(x)}{|\nabla\psi(x)|}\, dl = -\int_{G(z)} \frac{\partial\beta_3(x_1, x_2, x_3)}{\partial x_3}\, dx_1\, dx_2,$$

$$D(z, x_3) = \oint_{C(z)} \frac{\beta_3(x_1, x_2, x_3)}{|\nabla\psi(x_1, x_2)|}\, dl, \tag{4}$$

where $G(z) \subset R^2$ is the domain bounded by $C(z)$. We used here that $\operatorname{div} \beta(x) = 0$, so that $\frac{\partial\beta_1}{\partial x_1} + \frac{\partial\beta_2}{\partial x_2} = -\frac{\partial\beta_3}{\partial x_3}$. The limiting slow motion takes place in $[0, \infty) \times R^1$. It is easy to check that the boundary $\{O\} \times R_1$ of this half-plane is inaccessible in finite time for the limiting slow motion.

Let us now consider the case when the stream function $\psi(x_1, x_2)$ has more than one critical point (see Fig. 1), $\min \psi(x_1, x_2) = 0$. Denote by $\Gamma$ the graph homeomorphic to the set of connected components $C_i(z)$ of the level sets $C(z) = \{x \in R^2 : \psi(x) = z\} = \bigcup_i^{n(z)} C_i(z)$, $z \in [0, \infty)$, provided with the natural topology (compare with refs. 3 and 4).

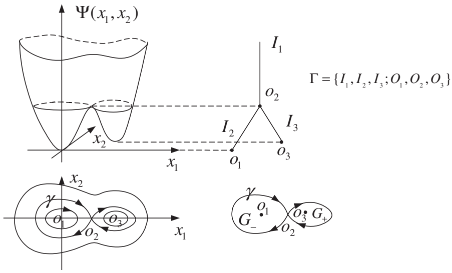


Figure 1.

Let $Y: R^3 \to \Gamma \times R^1 = \Pi$ be the projection of $R^3$ on $\Pi = \Gamma \times R^1$: $Y(x_1, x_2, x_3)$ is the point of $\Pi$ such that the second component equal to $x_3$ and the first component is the point of $\Gamma$ corresponding to the level set component of $\psi(x_1, x_2)$ containing the point $(x_1, x_2)$. Let us number the edges of $\Gamma$. Let the triple $(\psi(x), k(x), x_3)$ is the point of $\Pi$ corresponding to $x = (x_1, x_2, x_3)$ in such a mapping; $k(x)$ is the number of the edge of $\Gamma$ containing $Y(x)$. The slow component of the motion is now $Y_t = Y(X_t^\varepsilon)$. It changes in the set $\Pi = \Gamma \times R^1$ which is called open book. In the case of $\psi(x)$ with one saddle point like in Fig. 1, the set $\Pi$ is shown on Fig. 2.

Until $Y_t$ belongs to the same page of the open book $\Pi$, it can be described by standard averaging principle as above. The only difference is that, on each page, the averaging should be performed over the corresponding $C_i(z)$ or $G_i(z)$ so that the functions $B$ and $D$ in system (3) now depend on the number of the page (on the number of the edge of $\Gamma$); $B = B_i(z, x_3)$, $D = D_i(z, x_3)$, $T_i(z)$ are, in general different for different $i \in \{1, 2, 3\}$.

When a trajectory of the limiting slow motion approaches the set $Q_k = \{O_k\} \times R^1$, where $O_k$ is a vertex of $\Gamma$, $B_i(z, x_3)$ tends to zero. If

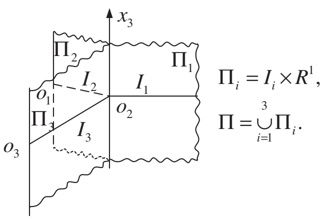


Figure 2.

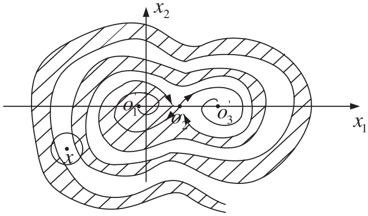


Figure 3.

$\psi(x_1, x_2)$ has at $O_k$ an extremum (such vertices of $\Gamma$ are called exterior), then the set $O_k \times R^1$ is inaccessible (see refs. 3 and 4). But it is easy to check that if $O_k$ corresponds to a saddle point of $\psi(x_1, x_2)$, the set $\{O_k\} \times R^1$ (the binding of the open book) can be accessible in a finite time due to the logarithmic grows of $T_i(z)$ as $z \to \psi(O_k)$.

As a result of the accessibility of the binding, the limit of the slow component $Y_t^\varepsilon$ as $\varepsilon \downarrow 0$ may not exist for large enough t. One can see this from Fig. 3, where the phase picture of the projection of the trajectories of the perturbed flow on the plane $(x_1, x_2)$ is shown for $\psi(x_1, x_2)$ as in Fig. 1 and perturbations $\beta(x) = (\beta_1(x_1, x_2), \beta_2(x_1, x_2), \beta_3(x_1, x_2, x_3), \frac{\partial \beta_3}{\partial x_3}(x) > 0$. The separatrices of the saddle point $O_2^{'}$ are shown in the picture. The width of the ribbons is of the order $\varepsilon$ as $\varepsilon \downarrow 0$. The shadowed ribbon enters the left domain $G_-$ (see Fig. 1). The white ribbon enters $G_+$. If $x$ is situated outside of the curve $\gamma$, it will alternatively belong to shadowed or to white ribbon as $\varepsilon \downarrow 0$. Since the slow component enters $G_- \cup G_+$ in a finite time, the limit of $(\psi(X_1^\varepsilon(t), X_2^\varepsilon(t)), k(X_1^\varepsilon(t), X_2^\varepsilon(t)))$ as $\varepsilon \downarrow 0$ does not exist for $t$ greater than the entrance time.

To regularize the problem and to establish the averaging principle in more or less general situation, one can introduce additional small stochastic perturbations of system (2). This can be done in various ways. The first way, and this, roughly speaking, is the traditional approach in the theory of dynamical systems, is to consider random perturbations of the initial conditions.

Let $\xi^\delta$ be the random variable distributed uniformly in the ball $\mathscr{E}^\delta = \{x \in R^3, |x| \leqslant \delta\}$. Let $X_t^{\varepsilon,\delta}$ be the solution of (2), with the initial condition $X_0^{\varepsilon,\delta} = x + \xi^\delta$. Then $X_t^{\varepsilon,\delta}$ and its projection $Y_t^{\varepsilon,\delta} = Y(X_t^{\varepsilon,\delta})$ on $\Pi$ are stochastic processes.

Consider a stochastic process $Y_t$, $0 \leqslant t \leqslant T$, on the open book $\Pi$ which is defined as follows:

Inside each page $\pi_i = I_i \times R^1$, $I_i \subset \Gamma$, the evolution of $Y_t$ is deterministic and is governed by the equations ($(\psi, x_3)$ are the coordinates in $\pi_i$)

$$\frac{d\psi(t)}{dt} = \frac{1}{T_i(\psi(t))} B_i(\psi(t), X_3(t)), \psi(0) = \psi(x_2, x_2),$$
$$\frac{dX_3(t)}{dt} = \frac{1}{T_i(\psi(t))} D_i(\psi(t), X_3(t)), X_3(0) = x_3. \tag{5}$$

The coefficients $B_i(z, x_3)$, $D_i(z, x_3)$, $T_i(z)$ are defined by formulas (4) with $C(z)$ and $G(z)$ replaced by $C_i(z)$ and $G_i(z)$ where $C_i(z)$ are the components of z-level set of $\psi(x_1, x_2)$ corresponding to $I_i \subset \Gamma$ and $G_i(z)$ are domains bounded by $C_i(z)$. Trajectory $Y_t = (\psi(t), x_3(t))$ can reach the binding (axis $x_3$ in Fig. 2.) of the book in a finite time. Let it be the line $Q_k = O_k \times R^1$, where $O_k$ is the vertex of $\Gamma$ corresponding to a saddle point of $\psi(x_1, x_2)$. Three pages are glued together at $Q_k$: $\pi_{i_1}, \pi_{i_2}, \pi_{i_3} \subset \Pi$. Let $Y_t$ come to a point $M \subseteq Q_k$ from the plane $\pi_{i_1}$. It follows from the definition of functions $B_i(z, x_3)$, that at least one of the fields $B_{i_2}(z, x_3)$ in $\pi_{i_2}$, or $B_{i_3}(z, x_3)$ in $\pi_{i_3}$ is directed from $Q_k$ in a vicinity of point M. If there is just one such field, say $B_{i_2}(z, x_3)$, the trajectory of $Y_t$ leaves M immediately to the plane $\pi_{i_2}$ and moves along the field $(B_{i_2}(z, x_3), D_{i_2}(z, x_3))$ in $\pi_{i_2}$. If both fields $B_{i_2}(z, x_3)$ and $B_{i_3}(z, x_3)$ are directed from $Q_k$, the process $Y_t$ also leaves $Q_k$ immediately and goes to $\pi_{i_2}$ or to $\pi_{i_3}$ with probabilities $P_2(M)$ and $P_3(M)$ independently of its behavior before it comes to the point $M \in Q_k$. To define the probabilities $P_2(M)$ and $P_3(M)$, consider the $\gamma$-curve (see Fig. 1), corresponding to the vertex $O_k$. Let the non-perturbed trajectories situated inside $G_-$ near $\gamma$ correspond to $I_{i_2}$, trajectories inside $G_+$ correspond to $I_{i_3}$ (and trajectories outside $G_- \cup G_+$ corresponds to $I_{i_1}$). Note, that, in general, the non-perturbed system may have inside $G_-$ or $G_+$ other saddle points. Let the point $M \in \Pi$ has coordinates $(z, i, x_3)$. Then

$$P_2(M) + P_3(M) = 1, \qquad \frac{P_2(M)}{P_3(M)} = \frac{\left| \int_{G_-} \frac{\partial \beta_3(x_1, x_2, x_3)}{\partial x_3} dx_1\, dx_2 \right|}{\left| \int_{G_+} \frac{\partial \beta_3(x_1, x_2, x_3)}{\partial x_3} dx_1\, dx_2 \right|}. \tag{6}$$

These conditions define $P_2(M)$ and $P_3(M)$.

The process $Y_t$ on $\Pi$ defined above is unique. The stochasticity of $Y_t$ is concentrated just on the binding of the open book, more precisely, at points of the binding such that there are two exits from them.

One can prove (compare with ref. 1), that if $\psi(x_1, x_2)$ has just one saddle point like in Fig. 1, the process $Y^{\varepsilon, \delta}(t)$ converge weakly in the space $C_{0T}(\Pi)$ of continuous on $[0, T]$ functions with values in $\Pi$ to the process $Y_t$, if, first, $\varepsilon \downarrow 0$ and then $\delta \downarrow 0$.

But it turns out, that if the stream function has more than one saddle point, the regularization by random perturbation of the initial point does not exist, at least, for some perturbations.

Therefore we will not go into details to prove the convergence of $Y_t^{\varepsilon,\delta}$ to $Y_t$ in the case of one saddle point.

One can use another way for the regularization of the problem: to perturb stochastically not the initial conditions but the equations themselves:

$$\dot{X}^{\varepsilon,\kappa}(t) = \frac{1}{\varepsilon}\,\overline{\nabla}\psi(X_1^{\varepsilon,\kappa}(t), X_2^{\varepsilon,\kappa}(t)) + \beta(X^{\varepsilon,\kappa}(t)) + \sqrt{\kappa}\,\sigma\dot{W}_t \tag{7}$$

Here $W_t = (W_t^1, W_t^2, W_t^3)$ is the Wiener process in $R^3$, $\sigma = (\sigma_{ij})$ is a non-degenerate $3\times 3$-matrix with constant coefficients, $0 < \kappa \ll 1$. As above, we consider $\overline{\nabla}\psi$ as 3-vector with $x_3$-component equal to zero. Define $a = (a_{ij}) = \sigma\sigma^*$. The generator of the diffusion process $X_t^{\varepsilon,\kappa}$ in $R^3$ has the form:

$$L^{\varepsilon,\kappa}u(x) = \frac{\kappa}{2}\,\mathrm{div}(a\nabla u(x)) + \beta(x)\cdot\nabla u(x) + \frac{1}{\varepsilon}\,\overline{\nabla}\psi(x)\cdot\nabla u(x).$$

The process $X^{\varepsilon,\kappa}(t)$ in $R^3$ has a fast and a slow components as $\varepsilon \downarrow 0$. The slow component is the projection of $X^{\varepsilon,\kappa}(t)$ on the open book $\Pi$: $Y_t^{\varepsilon,\kappa} = Y(X_t^{\varepsilon,\kappa})$.

It follows from ref. 5 that the family $Y_t^{\varepsilon,\kappa}$, $0 \leqslant t \leqslant T$, converges weakly as $\varepsilon \downarrow 0$ in the space $C_{0T}(\Pi)$ to a diffusion process $Y_t^\kappa$ on $\Pi$.

Inside a page $\pi_i \subset \Pi$, the process $Y_t^\kappa$ is governed by the operator

$$\bar{L}_i^\kappa u(z, x_3) = \frac{\kappa}{2T_i(z)}\left[\frac{\partial}{\partial z}\left(A_i^{11}(z)\,\frac{\partial u}{\partial z}\right) + A_i^{33}(z)\,\frac{\partial^2 u}{\partial x_3^2}\right] + \frac{1}{T_i(z)}\left[B_i(z, x_3)\,\frac{\partial U}{\partial z} + D_i(z, x_3)\,\frac{\partial u}{\partial x_3}\right].$$

The coefficients $B_i(z, x_3)$, $D_i(z, x_3)$, $T_i(z)$ were defined above; $A_i^{11}(z)$, $A_i^{33}$ are defined as follows:

$$A_i^{11}(z) = \int_{G_i(Z)} \mathrm{div}(a\nabla\psi(x_1, x_2))\,dx_1\,dx_2, \qquad A_i^{33} = a_{33}T_i(z).$$

Where $G_i(z)$ is the domain in the $(x_1, x_2)$-plane bounded by $C_i(z)$ such that $Y(G_i(z)\times R^1) \subset \pi_i$.

The operators $\bar{L}_i^\kappa$ define the process $Y_t^\kappa$ not in a unique way: one should add the gluing conditions at the binding of the open book $\Pi$. The gluing conditions define behavior $Y_t^\kappa$ after hitting the binding.

The gluing conditions can be described by the domain $D_A$ of the generator $A$ of $Y_t^\kappa$: a bounded continuous function $u(z, i, x_3)$ on $\Pi$ belongs to $D_A$ if

(1) inside each page $\pi_i \subset \Pi$, the function $u(z, i, x_3)$ has two continuous bounded derivatives;

(2) the function $Au$ (which is equal to $\bar{L}_i^\kappa u$ on $\pi_i$) is continuous on $\Pi$;

(3) at any line $Q_k = \{0_k\} \times R^1 \subset \Pi$ which binds pages $\pi_{i_1}$, $\pi_{i_2}$, $\pi_{i_3}$, $u(z, i, x_3)$ satisfies the condition

$$\sum_{l=1}^{3} \alpha_{i_l} D_l u(0_k, x_3) = 0 \qquad \text{for any} \quad x_3 \in R^1 \tag{8}$$

Here $D_l$ means differentiation in $z$ in the plane $\pi_{i_l}$, $l \in \{1, 2, 3\}$. To calculate the coefficients $\alpha_{i_l} = \alpha_{i_l}(0_k)$, consider the $\infty$-shaped curve $\gamma$ corresponding to the saddle point $O_k$ (see Fig. 1 and notations introduced there). Two bounded domains $G_-$ and $G_+$ are defined by $\gamma$. Let $I_{i_1}$ be the edge of $\Gamma$ corresponding to the periodic trajectories situated inside $G_-$ in a neighborhood of $\gamma$, $I_{i_2}$ corresponds to the trajectories inside $G_+$ which are close to $\gamma$, and $I_{i_3}$ corresponds to the exterior of $G_- \cup G_+$ near $\gamma$. Then

$$\alpha_{i_1} = \int_{G_-} \operatorname{div}(a\nabla\psi(x))\, dx_1\, dx_2, \qquad \alpha_{i_2} = \int_{G_+} \operatorname{div}(a\nabla\psi(x_1, x_2))\, dx_1\, dx_2, \tag{9}$$
$$\alpha_{i_3} = -(\alpha_{i_1} + \alpha_{i_2})$$

**Remark.** Note that the gluing conditions (8), (9) for the process $Y_t^\kappa$ on $\Pi$ are the same for all $x_3$ and for all $\kappa$, and they are independent of the deterministic perturbation $\beta(x_1, x_2, x_3)$. If the diffusion matrix $a = \sigma\sigma^*$ is unit, then the coefficients $\alpha_i$ are equal to vorticity in corresponding domains: vorticity of the non-perturbed flow $\omega(x_1, x_2) = -\Delta\psi(x_1, x_2)$, and

$$\alpha_{i_1/i_2} = \left| \int_{G_{-/+}} \omega(x)\, dx \right|, \qquad \alpha_{i_3} = -\left| \int_{G_- \cup G_+} \omega(x)\, dx \right|.$$

Let us consider now the limiting behavior of $Y_t^\kappa$, $0 \leqslant t \leqslant T$, as $\kappa \downarrow 0$. By standard arguments one can check that the family $Y_t^\kappa$, $0 < \kappa \leqslant 1$, is tight in the weak topology in the space $C_{0,T}(\Pi)$ (compare with refs. 1 and 3.) Thus, to prove the convergence we should establish uniqueness of the limiting point.

It is easy to see that inside each page $\pi \in \Pi$ the limiting process $Y_t$, $0 \leqslant t \leqslant T$, is the deterministic motion governed by the Eqs. (5). For example, this can be proved using stochastic differential equations for process $Y_t^\kappa$ inside each page (see ref. 2, Chap. 4). Therefore, to prove the uniqueness, it is sufficient to check that the limiting process spends time zero at the binding and that the exit probabilities from any point of the binding are defined in a unique way.

In the case $\beta_1(x_1, x_2)$, $\beta_2(x_1, x_2)$ independent of $x_3$, both these statements follow from Lemmas 2.3 and 2.2 of ref. 1 respectively. Moreover, the exit probabilities from a point of the binding, in this case, according to ref. 1, are the same for all $x_3$, and they are defined by equalities (6). The case of perturbations depending on all variables $x_1, x_2, x_3$ can be reduced to the $x_3$-independent case by using comparison arguments.

Note, that in spite of the fact that the gluing conditions for $y_t^\kappa$, $\kappa > 0$, are independent of the perturbations $\beta(x)$ and defined just by the diffusion matrix $(a_{ij})$ (and, of course, the stream function), the exit probabilities for $Y_t$ are independent of $(a_{ij})$ and are defined by the deterministic perturbations $\beta(x)$. This means that stochasticity of the limiting slow motion of the flow is an intrinsic property of the non-perturbed system and its deterministic perturbations. The stochastic term serves just for regularization of the problem.

Combining these arguments, we have the following result:

**Theorem.** The slow component $Y_t^\varepsilon = Y(X_t^\varepsilon)$ of perturbed flow (2) converges as $\varepsilon \downarrow 0$ to the stochastic process $Y_t$ on the open book $\Pi$, which is defined by Eqs. (5) inside the pages and by conditions (8) at the binding. This means that the double limit of $Y(X_t^{\varepsilon,\kappa})$, where $X_t^{\varepsilon,\kappa}$ is defined by (7), as, first, $\varepsilon \downarrow 0$, and then $\kappa \downarrow 0$, in the weak topology in $C_{0T}(\Pi)$ is equal to $Y_t$. The limiting process is independent of stochastic perturbations, which were introduced for regularization of the problem: The double limit of $Y_t^{\varepsilon,\kappa}$ is the same for any matrix $\sigma$, $\det \sigma \neq 0$.

Finally, I will mention one more way of regularization of problem (2): Replace $\beta(x)$ in this equation by $\beta^\mu(x) = \beta(x) + \mu\tilde{\beta}(x)$, where $\tilde{\beta}(x)$ is a random field in $R^3$ and $\mu \in R^1$. Let $X_t^{\varepsilon,\mu}$ be the solution of Eq. (2) with such replacement. Under some assumptions concerning the random field $\tilde{\beta}(x)$ (compare with ref. 6), one can prove that the double limit $\lim_{\mu\downarrow 0} \lim_{\varepsilon\downarrow 0} Y(X_t^{\varepsilon,\mu})$ in the weak topology of $C_{0T}(\Pi)$ exits and equal to $Y_t$. So that this regularization again leads to the same process $Y_t$.

The intrinsic Stochasticity of incompressible and compressible flows were studied in a number of recent papers (see, for example, refs. 7 and 9, and a review paper 8). Stochasticity in those papers, as a rule, is the result of non-smoothness of the velocity field. If the velocity field is not Lipschitz, the differential equations for the trajectories may have non-unique solution. If we regularize the problem by addition of a small white noise $\sqrt{\kappa}\,\dot{W}_t$, $0<\kappa\ll 1$, corresponding stochastic equation has a unique solution. If now $\kappa\to 0$, we can have a stochastic process in the limit. One can consider other types of regularization as well. In our case, the intrinsic stochasticity on long time intervals is caused by the instabilities of the original non-perturbed flow. The distribution of the limiting slow motion will be the same for various types of the regularization, so that the stochasticity is intrinsic in a strong sense.

Finally, I will mention, that the weak compressibility of the fluid can lead to stochasticity of the stationary 2D-flow: weak compressibility means that $\operatorname{div} b(x)=\kappa\beta(x)=0$, $0<\kappa\ll 1$. Since $\beta(x)\not\equiv 0$, the slow component of the 2D-flow can reach the $\gamma$-curve of a saddle point of the stream function $\psi(x)$, $x\in R^2$, and then it may go with positive probabilities to one of the wells related to this saddle point.[(1)] Stochasticity related to week compressibility is studied also in ref. 7.

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